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On the Schild action for D = 0 and D = 1 strings

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Abstract

It is shown that the integration measure over the matrix Y in the matrix representation of the Schild action can be fixed by comparing the Schild matrix model with the random lattice string model for D=0. It is further checked that the given measure is consistent with the case D=1 as well.

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1 Introduction

Since Polyakov's seminal paper [1] on Quantum geometry of bosonic strings, there has been a tremendous effort to solve string theory exactly. Unfortunately, it has been realised that Polyakov's continuous approach does not take us any further than a perturbative description. For this reason other alternative formulations of strings have recently attracted a great deal of attention. Among them, Schild's action [2] seems to be the most promising candidate for a nonperturbative description of string theory. A remarkable fact is that both Polyakov's and Schild's actions can be derived from one theory [3]. In this sense these two models are dual to each other even though their field contents are different off-shell. It is argued that the given duality can be an exact equivalence at the quantum level [3].

A dramatic breakthrough occured when it was realised that the Schild action of type IIB superstring allows a natural matrix representation in the large N-limit[4]. For early ideas about the relation between the Schild string and matrix models see [5],[6] which also can be connected to a proposal on quantisation of the Schild action in ref. [7]. However, the IKKT matrix approach to the Schild formulation of (super)strings suffers one subtlety concerning the limit $N \to \infty$. Namely, in the IKKT matrix theory at the quantum level the size N of matrices is treated as a dynamical variable [4]. This seems to contradict the main idea of taking the large N limit in the matrix model. The mentioned subtlety has been cured in a modified matrix model proposed in [8] where a new matrix variable Y has been introduced so that the limit $N \to \infty$ can be taken straightforwardly. The curious fact is that Y does not couple to the fermionic matrices. Therefore, it appears that some of its properties can be studied in a pure bosonic theory.

The new matrix description suggested in [4],[8] provides us with a new tool of investigating nonperturbative properties of string theory. The hope is that these new matrix models can overcome the problems of the old matrix model approach to nonperturbative quantisation of strings [9] broadly studied a few years ago. The remarkable success of the old matrix models was a nonperturbative description of strings in dimensions $D \leq 1$. The major difficulties occured in an attempt to extend this approach to string theories in

D > 1 and superstrings.

If correct the new matrix models have to solve the given problems and at the same time they have to reproduce the exact results obtained for $D \leq 1$ strings within the old approach. The consistency between the new and the old matrix theories is required for justification of the equivalence between the Polyakov and the Schild formulations of string theory as suggested in [3]. This poses a question about the relation between these two matrix model approaches which has already been raised in [6]. One nontrivial result of such a relation has been discussed in [10].

The aim of this paper is to compare the new matrix model with the old matrix model description of the bosonic string in dimensions D = 0 and D = 1. By doing so we clarify some issues concerning the modified matrix theory constructed in [8]. In particular, we shall focus on the form of the potential for the matrix Y which cannot be fixed from the quasiclassical consideration. This potential defines the measure of the matrix integral over Y [12]. From this point of view our paper is a further extension of the analysis started in [12].

The paper is organized as follows. In section 2 we discuss the classical relation between Polyakov's and Schild's formulations of the Nabu-Goto string. Also we summarize the main ideas of introducing the matrix representation for the Schild action and implications of this representation for boundary conditions of the continuous theory. In section 3 we continue to study the quantization of the D=0 Schild action in the matrix representation. We show that the comparison with the D=0 Polyakov string allows us to fix the form of the potential of the matrix Y. This potential is further checked in the case of D=1 string. We conclude in section 4.

2 From strings to matrices

There exists an infinite number of classical functionals all of which give rise to one and the same equations of motion of strings. However, among them two formulations are distinguished by their simplicity and elegance. One is due to Polyakov [1] and the other due to Schild [2].

The Polyakov formulation is based on the following elegant action

$$S_{Polyakov} = -\frac{1}{\kappa} \int d^2 \xi \sqrt{-g} g^{ab} \partial_a X \cdot \partial_b X, \qquad (2.1)$$

where X^{μ} , $\mu = 1, 2, ..., D$, are coordinates of the string in the target space-time and g^{ab} is the two-dimensional metric on the string world sheet parametrized by ξ^1 and ξ^2 . $\kappa = 4\pi\alpha'$ is a constant related to the string tension.

The Schild action is defined as follows

$$S_{Schild} = -\int d^2\xi \left(-\frac{1}{2\kappa^2 e} \Sigma^2 + e \right). \tag{2.2}$$

Here

$$\Sigma^{\mu\nu} = \epsilon^{ab} \partial_a X^{\mu} \partial_b X^{\nu}. \tag{2.3}$$

The variable e is a positive definite scalar density defined on the world sheet.

It is easy to check that by excluding the metric g^{ab} in the Polyakov action and the density e in the Schild action one arrives at the standard Nabu-Goto action. Moreover the two models are dual to each other [3]. Indeed, let us consider the following functional

$$S(t, e, X) = \int d^2 \xi \frac{1}{e} \left(\det t + \frac{1}{\kappa} t^{ab} \partial_a X \cdot \partial_b X \right) - \int d^2 \xi \ e, \tag{2.4}$$

where the new variable t^{ab} was introduced.

The equivalence of S(t, e, X) to the Schild action follows from the identity

$$S(t, e, X) = S_{Schild}(e, X) + \int d^2 \xi \frac{1}{e} \det \tilde{t}, \qquad (2.5)$$

where

$$\tilde{t}^{ab} = t^{ab} + \frac{1}{\kappa} \epsilon^{ac} \epsilon^{bd} \partial_c X \cdot \partial_d X. \tag{2.6}$$

Now it is easy to see that upon the equation of motion of \tilde{t}^{ab} , which is

$$\tilde{t}^{ab} = 0, (2.7)$$

the functional S(t, e, X) reduces to the Schild action (2.2) of e and X^{μ} .

The equivalence to the Polyakov action is also straightforward. It is convenient to make a change of variables:

$$t^{ab} = e^2 g^{ab},$$

$$e = \tilde{e}\sqrt{-g}.$$
(2.8)

Then

$$S(t, e, X) \to S(\tilde{e}, g^{ab}, X) = \int d^2\xi \ \tilde{e}(\tilde{e}^2 - 1)\sqrt{-g} + \frac{1}{\kappa} \int d^2\xi \ \tilde{e}\sqrt{-g}g^{ab}\partial_a X \cdot \partial_b X. \quad (2.9)$$

Correspondingly, the equation of motion of g^{ab} is given as follows

$$\frac{1}{2}\tilde{e}(\tilde{e}^2 - 1)g_{ab} + \frac{1}{\kappa} \left(\partial_a X \cdot \partial_b X - \frac{1}{2}g_{ab}g^{cd}\partial_c X \cdot \partial_d X \right) = 0. \tag{2.10}$$

Take the trace of the given equation of motion. We get

$$\tilde{e}(\tilde{e}^2 - 1) = 0, (2.11)$$

which implies

$$\tilde{e} = 1, \tag{2.12}$$

since we assumed that e > 0.

Thus, if we plug solution (2.12) back into formula (2.9), we obtain the Polyakov action for g^{ab} and X^{μ} . This completes the prove of the classical equivalence between the Polyakov and the Schild formulations.

At the quantum level the situation is much more complex*. The formal manipulations under the functional integral discussed in [3] are probably correct only for the critical strings. For noncritical strings the analysis gets stuck because of lack of a nonperturbative definition of the functional measure. However, as we shall see in some cases the comparison of the two models can be done even in noncritical dimensions. These will be the cases when both the theories are represented as matrix models.

Now we want to turn to the matrix description of the Schild action. It starts with an observation that the latter takes a pure algebraic form in terms of the following Poisson brackets [7],[6]

$$\{A, B\} \equiv \frac{1}{e} \epsilon^{ab} \partial_a A \partial_b B. \tag{2.13}$$

The next drastic step is to replace all Poisson brackets by commutators of $N \times N$ matrices [4] which is justifiable only in the limit $N \to \infty$. A subtlety arises in passing from the two *The quantum equivalence between the Nabu-Goto action and the Schild action has been studied in [13].

dimensional integral over the world sheet to the matrix trace. In ref.[4] this transition is defined according to the following rule

$$\{A, B\} \rightarrow [A, B],$$

$$\int d^2 \xi \ e \rightarrow \text{Tr.}$$
 (2.14)

With the given rule the Schild action is replaced by the following matrix model [4]

$$S_{IKKT} = -\frac{\alpha}{4} \text{Tr}[X^{\mu}, X^{\nu}]^2 + \beta \text{Tr} \mathbf{1},$$
 (2.15)

where α and β are some constants of order N. As one can see this matrix model does not have a matrix variable for the density e of the continuous theory. According to [4] the size N of matrices has to be treated as such an additional dynamical degree of freedom.

In what follows we shall study a modified matrix model [8] which we shall call Schild matrix model (SMM). Its classical action is given as follows

$$S_{SMM} = -\frac{\alpha}{4} \text{Tr } Y^{-1} [X^{\mu}, X^{\nu}]^2 + \beta \text{Tr } Y, \qquad (2.16)$$

where Y is a matrix variable associated with the field e. The given matrix theory can be derived from the Schild action by the following rule [8]

$$e \rightarrow Y,$$

$$\{X^{\mu}, X^{\nu}\}_{P.B.} \equiv \epsilon^{ab} \partial_a X^{\mu} \partial_b X^{\nu} \rightarrow [X^{\mu}, X^{\nu}]$$

$$\int d^2 \xi \rightarrow \text{Tr.}$$

$$(2.17)$$

This rule (2.17) contains some nontrivial information about boundary conditions of the world sheet. Indeed, according to the standard property of the trace one can derive the following relation

$$Tr[X^{\mu}, X^{\nu}] = 0.$$
 (2.18)

In terms of continuous strings this means

$$\int d^2\xi \ \{X^{\mu}, X^{\nu}\}_{P.B.} = \int d^2 \ \epsilon^{ab} \partial_a X^{\mu} \partial_b X^{\nu} = \int_{\Gamma} ds \ X^{\mu} \partial_t X^{\nu} = 0, \tag{2.19}$$

where ∂_t is a derivative tangential to the boundary Γ . Eq.(2.19) is fulfilled when X^{μ} obeys the following boundary condition

$$\partial_t X^{\mu}|_{\Gamma} = 0 \tag{2.20}$$

In other words, if there are boundaries on the world sheet, all X^{μ} 's have to satisfy the Dirichlet boundary conditions. This observation supports a conjecture that type IIB matrix model (whose bosonic part is described by the SMM) is the effective action of N D-instantons [11],[4]. It is even more interesting to look at the D-brane solutions of the (super)-SMM which are given as follows [4]

$$[B^{\alpha}, B^{\beta}] = -ig^{\alpha\beta} \mathbf{1}, \tag{2.21}$$

where B^{α} , $\alpha = 0, 1, ..., p$, (for odd p) are world volume coordinates of a p-brane [4],[8] and $g^{\alpha\beta}$ is some antisymmetric matrix. The rest of the coordinates can be set to zero. Solutions of this type make sense only in the large N limit. In terms of the Poisson brackets eq.(2.21) is equivalent to the following relation

$$\int d^2\xi \ \{B^{\alpha}, B^{\beta}\}_{P.B.} = \int_{\Gamma} ds \ B^{\alpha} \partial_t B^{\beta} \neq 0. \tag{2.22}$$

In order for this to be the case, B^{α} have to satisfy the following boundary condition

$$\partial_n B^\alpha |_{\Gamma} = 0, \tag{2.23}$$

where ∂_n is a derivative normal to the boundary. Indeed, it is well known that the only natural boundary conditions in string theory are either Neumann or Dirichlet conditions [14]. Since $\partial_t B^{\alpha} \neq 0$ on the boundary, condition (2.23) has to be imposed. This is nothing but the Neumann boundary condition as it should be in the case of D-branes [15]. All other coordinates still have to obey the Dirichlet boundary conditions. Moreover, eq.(2.22) signals that the solutions of type (2.21) require the presence of boundaries on the world sheet. Thus, these solutions can be understood as a nonperturbative effect of the string interaction: boundaries are generated in the process of multi-string coupling. Multi-string states are inevitable ingredients of the given matrix description due to the claster decomposition of matrices X^{μ} .

3 D=0 and D=1 SMM

The matrix representation of the Schild action allows us to define a nonperturbative quantization of strings in terms of large N matrix integrals [4],[8]. This is a nontrivial task because the integration over the matrix Y requires a properly defined measure which cannot be obtained in the quasiclassical approximation. The measure can be understood as a potential V(Y) in the SMM [12]. Therefore, we write down the following general expression for the quantum SMM

$$S_{SMM} = N \left[\text{Tr } Y^{-1} \Lambda + V(Y) \right], \tag{3.24}$$

where

$$\Lambda \sim [X^{\mu}, X^{\nu}]^2. \tag{3.25}$$

The form of the potential V(Y) has been discussed in [8],[12]. In [12] it is argued that the potential V(Y) has to be fixed as follows

$$V_{FMOSZ-CZ}(Y) = \beta \text{Tr} Y + \gamma \text{Tr} \ln Y, \qquad (3.26)$$

where it is essential to choose the parameter γ of order one [12]:

$$\gamma = 1 + \mathcal{O}(1/N). \tag{3.27}$$

The argument is based on the locality and reparametrisation invariance of the corresponding effective action (obtained by integrating out the matrix Y).

In what follows we would like to put forward more arguments in favour of a further modification of the potential V(Y). To this end we would like to turn to two particular cases of D = 0 and D = 1 SMM's.

Let us start with the D=0 SMM. Because there are no space-time coordinates X^{μ} , the corresponding partition function is completely defined by the potential V(Y):

$$Z_{SMM}(D=0) = \int \mathcal{D}Y \exp[-N V(Y)], \qquad (3.28)$$

where $\mathcal{D}Y$ is the standard flat measure. This partition function is supposed to give rise to the nonperturbative expression for free energy.

At the same time, following the random matrix representation of the D=0 Polyakov string [16],[17], one can write down the explicit expression for the corresponding D=0 free energy in terms of the integral over a hermitian matrix

$$Z_P(D=0) = \int \mathcal{D}M \, \exp\left[-N \, \text{Tr}\left(\frac{1}{2}M^2 - gM^4\right)\right],\tag{3.29}$$

where the coupling constant g approaches the critical value

$$g \to g_c = \frac{1}{48}.$$
 (3.30)

In order to compare $Z_P(D=0)$ with $Z_{SMM}(D=0)$, we present the former in the following equivalent form [16]

$$Z_P(D=0) = const \int \mathcal{D}U \exp\left\{-N \operatorname{Tr}\left[\frac{1}{2}U^2 + \ln(1-\lambda U)\right]\right\}, \qquad (3.31)$$

where

$$\lambda = \sqrt{8g}.\tag{3.32}$$

The important point to be made is that the given matrix model possesses a remarkable symmetry known as the face/vertex duality of the Feynmann graphs [16],[18] which is related to the T-duality of the continuous string theory [19],[20].

It is convenient to further change variables

$$1 - \lambda U \to \lambda^2 Y,\tag{3.33}$$

in terms of which the potential in eq.(3.31) takes the following form

$$V(Y) = N \operatorname{Tr} \left[-Y + \ln Y + \frac{\lambda^2}{2} Y^2 \right] + \delta, \tag{3.34}$$

where

$$\delta = N^2 \left(\frac{1}{2\lambda^2} + \ln \lambda^2 \right). \tag{3.35}$$

The first two terms in eq.(3.34) are nothing but the Penner model at the critical point [21],[22]

$$V_{Penner}(Y) = Nt \operatorname{Tr} \left(\ln Y - Y \right), \tag{3.36}$$

where

$$t \to t_c = 1. \tag{3.37}$$

This part of the potential V(Y) is in agreement with the expression proposed in ref.[8] and given by eq.(3.26). However, there is the Gaussian term in eq.(3.34) which can be understood as a correction to formula (3.26) in the limit $\lambda \to 0$ (as advocated in [23]).

As one can see the equivalence between the Polyakov D = 0 string and the D = 0 SMM requires the modified potential given by eq.(3.34). The nature of the corrected potential can be further clarified in the case of D = 1 string.

Remarkably, the D=1 SMM is described like the D=0 case by the potential alone[†]:

$$S_{SMM}(D=1) = N V(Y).$$
 (3.38)

Indeed, the matrix Λ in eq.(3.24) vanishes for a single X,

$$\Lambda \sim [X, X]^2 = 0. \tag{3.39}$$

Thus the D=1 SMM partition function is defined according to

$$Z_{SMM}(D=1) = \int \mathcal{D}X \int \mathcal{D}Y e^{-NV(Y)}.$$
 (3.40)

It differs from formula (3.28) by the additional integral over the matrix X. Let us introduce the following notation

$$\nu \equiv \int \mathcal{D}X. \tag{3.41}$$

Then the partition function can be presented as follows

$$Z_{SMM}(D=1) = \nu e^{-\delta} \int \mathcal{D}Y \exp\left[-N \operatorname{Tr}\left(\ln Y - Y + \frac{\lambda^2}{2}Y^2\right)\right], \qquad (3.42)$$

where δ is given by eq.(3.35).

Let us take the limit

$$\lambda \to 0.$$
 (3.43)

It is easy to see that in this limit the constant δ behaves as follows

$$\delta \to \frac{N^2}{2\lambda^2}.\tag{3.44}$$

[†]It might be interesting to compare the given similarity between the D=0 and D=1 descriptions and the phase transition between the c=0 and c=1 CFT's coupled to 2D gravity discovered in [20],[24]

Consider the following integral

$$Z(\eta) = e^{-\frac{N \operatorname{Tr} \eta^2}{2}} \int \mathcal{D}Y \exp \left[-N \operatorname{Tr} \left(\ln Y - Y + \frac{1}{2} \eta^{-1} Y \eta^{-1} Y \right) \right], \tag{3.45}$$

where

$$\eta = \operatorname{diag}(1/\lambda, 1/\lambda, ..., 1/\lambda). \tag{3.46}$$

Obviously,

$$Z_{SMM}(D=1) = \nu Z(\eta). \tag{3.47}$$

The beautiful thing is that $Z(\eta)$ satisfies the Schwinger-Dyson equation [25]

$$\left(\frac{\partial^2}{\partial \eta^2} + N\eta \frac{\partial}{\partial \eta} - N^2\right) Z(\eta) = 0, \tag{3.48}$$

which is equivalent to the Virasoro constraints of the generalized Kontsevich-Penner model (or the Gaussian Kontsevich model) [25],[26]. Correspondingly, eq.(3.46) has to be thought of as being a constraint on solutions of eq.(3.48) with the arbitrary matrix η . Such solutions do exist [25],[26]. This model has been extensively studied in [25],[26],[27], see also [28]. It has been shown that in the limit $\lambda \to 0$ this theory coincides with the standard Penner model. The latter is known to describe the c=1 CFT (Polyakov's D=1 string) compactified on a circle of the self-dual radius [22],[29]. It is remarkable that exactly at the self-dual radius the quantity ν in eq.(3.41) is equal to one.

All in all we arrive at the conclusion that the Polyakov D=1 string and the D=1 SMM are equivalent if we choose the potential V(Y) in form (3.34). Moreover, the equivalence requires the specific self-dual value of the radius of the compact dimension. The latter may imply that a D=1 string does not exist at different values of the compactification.

4 Conclusion

We have shown that a simple consideration of the SMM at D=0 allows us to fix the potential V(Y) in the following form

$$V(Y) = N \operatorname{Tr} \left[\frac{Y^2}{2} + \ln(1 - \lambda Y) \right]. \tag{4.49}$$

This matrix theory is known as the Gaussian Kontsevich model. It is invariant under the face/vertex duality which seems to play a significant role in the nonperturbative understanding of string theory. The given potential modifies the anzats suggested in [8] when λ is not vanishing. The critical value of the matrix model coupling constant λ depends on the dimensionality of X^{μ} 's. It is very interesting to understand how the constant λ is related to the string coupling constant g_s . One plausible conjecture is that

$$\lambda = \xi + \mathcal{O}(\xi^2), \tag{4.50}$$

where

$$\xi = \frac{g_s}{1 + g_s^2}. (4.51)$$

In this case, the SMM is consistent with the symmetry $g_s \to 1/g_s$ which is thought of as being a nonperturbative symmetry of IIB superstring [30],[31].

In dimensions D > 1 the SMM is formulated as follows

$$S_{SMM} = N \operatorname{Tr} \left[(1 - \lambda Y)^{-1} \Lambda + \frac{Y^2}{2} + \ln(1 - \lambda Y) \right].$$
 (4.52)

This matrix model deserves further investigation.

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